Evaluating Welfare with Nonlinear Prices

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Abstract

Economists frequently evaluate the welfare consequences of nonlinear incentives and prices, ranging from the effects of price discrimination by firms to the benefits of many public-sector transfer programs. This paper extends existing methods to accommodate a broad range of modern pricing practices, including menus of pricing plans. The techniques we present are straightforward to implement in applied work with both parametric and nonparametric demand models, and do not require analytic representations for preferences.

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1 Introduction

Economists frequently evaluate the benefits and costs of price and tax changes. Two important papers in the early 1980s, by Jerry Hausman (1981) and Yrjö Vartia (1983), facilitated these evaluations by showing how to use estimates of ordinary (Marshallian) demand functions to calculate exact (Hicksian) consumer welfare and deadweight loss. These techniques have been widely applied to study the welfare consequences of policy changes in public finance, labor economics, industrial organization, and international trade.[1]

Evaluating exact consumer welfare and deadweight loss poses a more difficult problem if the consumer faces a nonlinear budget constraint. Leading applications include nonlinear tax schedules in public finance, and the consequences of price discrimination using 'menus' of pricing plans in product markets.[2] Although some extensions of Hausman's (1981) observations have been applied in these settings, existing methods for evaluating exact consumer welfare changes remain quite restrictive. They require analytic representations for both demand and (indirect) utility and, in application, have been limited to a handful of demand specifications for which these representations are easy to manipulate.

In this paper, we show how to evaluate exact consumer welfare in more general settings. Paralleling Vartia (1983), our principal contribution is to address how to evaluate exact consumer welfre for general nonlinear price changes when the researcher does not use a demand model with a simple, analytic indirect utility representation. This is particularly useful in applications where demand is estimated using semi- or non-parametric econometric techniques and analytic utility functions cannot be determined. Second, our approach is not restricted to economic settings with convex, piece-wise linear budget constraints; rather, it readily accommodates non-convex sets and non-piecewise linear price schedules. Finally, our exposition simplifies prior treatments. We show that the relevant calculations have an elegant supply and demand representation, which clarifies the connection between Hicks' exact consumer welfare measures for general nonlinear and classical (linear) budget sets. We also indicate computational simplifications to prior methods.

2 A Graphical Analysis

Our central insights are simplest to convey in terms of supply and demand, rather than a traditional budget set diagram. Figure 1 is a standard two-good depiction of the com-

[1] See Becht (1995) and Slesnick (1998) for surveys.

[2] Early applications include changes in tax rates or subsidies that vary with income or consumption (Burtless and Hausman 1978, Blomquist 1983, Hausman 1985, Blundell, Preston and Walker 1994). Recent applications involve the design of self-selecting 'menus' of prices, which are increasingly prevalent for an array of insurance, energy, travel, communication, entertainment, and digital information services (Wilson 1993, Miravete and Roller 2003, Rysman 2004, Reiss and White 2005).

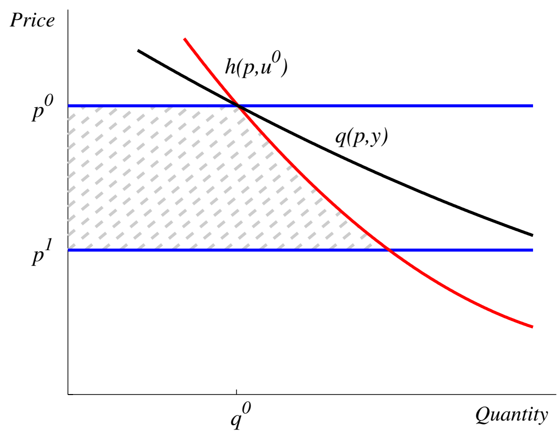

Figure 1

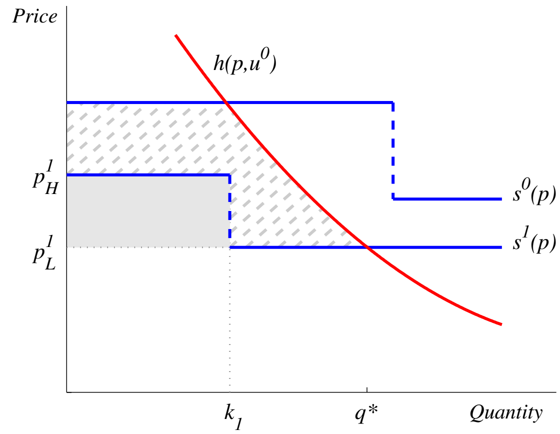

Figure 2

pensating variation (CV) if relative prices change from p^0 to p^1. In this diagram $q(p, y)$ represents the ordinary (Marshallian) demand curve and $h(p, u^0)$ the compensated (Hicksian) demand curve, assuming the consumer faces a constant price p and has income y. As is well known, the compensating variation equals the dashed region under the Hicksian demand curve between p^0 and p^1. This integral is straightforward to obtain from the indirect utility function (Hausman, 1981) or the demand function alone (Vartia, 1983).

Figure 2 illustrates the compensating variation for a nonlinear price change, here in the form of a volume discount. For convenience, we refer to the nonlinear price schedules as "supply" schedules, and label the initial and new supply schedules $s^0(p)$ and $s^1(p)$ respectively. In Figure 2 the compensated demand curve intersects the new supply schedule at its lower marginal price, p^1_L. Unlike with constant marginal prices, the compensating variation is not simply the area under compensated demand between the initial and new (marginal) prices. Instead, allowance must be made for the fact that the consumer pays a premium of $d^1 = (p^1_H - p^1_L)k_1$ for the first k_1 units, where k_1 is the quantity at which the marginal price drops from p^1_H to p^1_L. In Figure 2 the compensating variation is the area under h between the initial and new marginal prices, less the shaded grey region (d^1). We refer to $d^i(p)$ as the *infra-marginal discount* (or *premium*) associated with supply schedule $s^i(p)$.[3]

Figure 2 appears to suggest that piecewise-linear price schedules pose few additional complications. However, Figure 3 shows a more complicated case where (*i*) the new supply schedule overlaps the initial supply schedule, and (*ii*) the compensated demand curve crosses the new supply schedule at more than one point. The overlap in the supply schedules makes the sign of the compensating variation unclear *a priori*, and multiple crossings

[3]Sign conventions vary in the literature; we use $CV > 0$ if the consumer becomes better off, in the precise sense that initial income exceeds the minimum income that compensates for a price change.

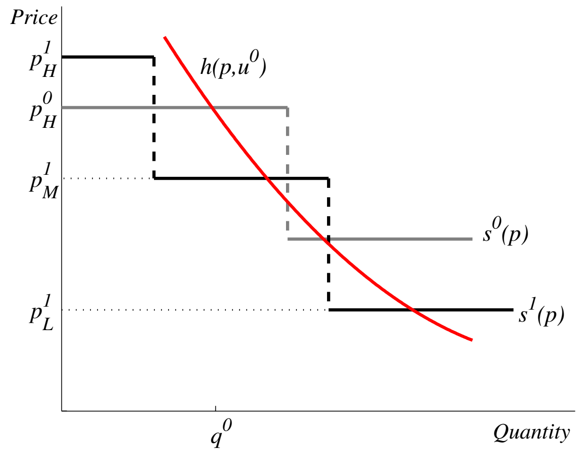

Figure 3

pose the problem of which marginal price is appropriate for calculating it.

To determine the compensating variation with a nonlinear price change, we must evaluate which price is marginal at the minimum income that leaves the consumer indifferent *ex post*. Here the compensated demand curve intersects the new supply schedule $s^1(p)$ at two possible marginal prices, p_M^1 and p_L^1.[4] Consider first p_M^1. In Figure 4(a), the area under h between p_M^1 and the initial marginal price p_H^0 is $B + C$. From this we deduct the new infra-marginal premium, or area $A + B$. This yields a candidate (local) compensating variation value of $(B + C) - (A + B) = C - A$ with price p_M^1 at the margin.

We must also evaluate other points at which supply equals compensated demand because they may result in a smaller compensating income. In this example, the CV corresponding to p_L^1 equals the sum of the four shaded regions in Figure 4(b), each of which is labeled with its respective sign. These regions are determined by the area under h between the candidate marginal price p_L^1 and the initial marginal price p_H^0, less the new infra-marginal premium of $k_1^1(p_H^1 - p_M^1) + k_2^1(p_M^1 - p_L^1)$.

In this example there are two candidate values of CV. However, one of the two candidate CV values is too generous; the correct one provides the lowest compensated income, $y - CV$, that achieves u^0. Thus we take the larger of the two candidate CV values. (In budget-set space, the correct value of CV yields a compensated budget set nowhere above the initial indifference curve). The correct value is easy to determine here from Figure 4(b): It is the value of CV associated with the lower marginal price, which is greater than the candidate value CV at the middle price.

This discussion indicates some of the issues that arise when evaluating exact consumer

[4] In budget-set space, this implies there are two (distinct) quantities at which the slope of the new budget constraint equals the marginal rate of substitution along the initial indifference curve.

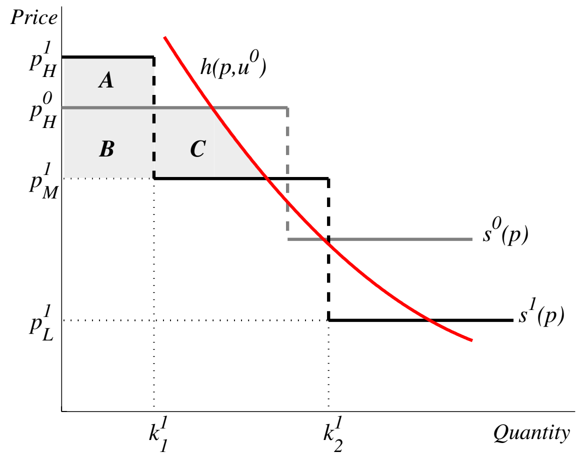

Figure 4(a)

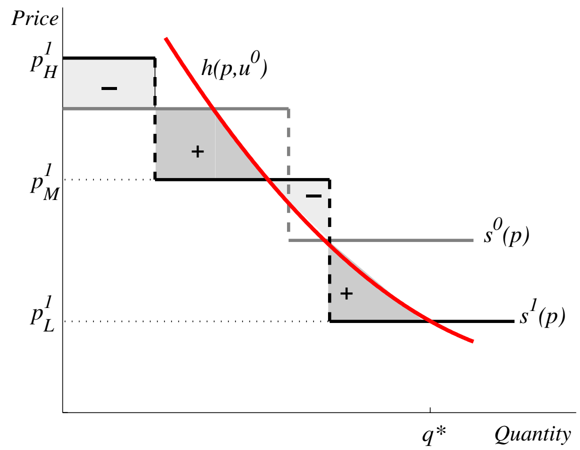

Figure 4(b)

welfare with nonlinear prices. It remains to show how this can be done without an analytic representation for the compensated demand function, h, and when the budget constraint is more complex than the simple piece-wise linear case illustrated here. Before addressing more general methods, however, it is useful to connect these ideas to prior methods in the literature.

3 Known Indirect Utility

Hausman (1981) and Vartia (1983) proposed methods for calculating exact consumer welfare assuming the consumer faces a linear budget constraint. Later extensions considered how to proceed if the budget constraint is piece-wise linear. We summarize this procedure here to highlight some important complications and lay the groundwork for its generalization.

Informational requirements. Prior techniques for evaluating exact consumer welfare with nonlinear prices proceed under two conditions. First, the researcher knows the consumer's indirect utility function $v(p, y)$ and ordinary demand function $q(p, y)$ that would apply if the consumer faced constant (relative) prices p that do not depend on q. Second, the researcher observes initial income and prices (or consumption q^0), and can therefore evaluate initial utility u^0.

In theory, the first condition is unrestrictive because v can be obtained from q as the solution to a differential equation via Roy's Identity. In practice, however, this can be difficult and there are only a few demand models for which an analytic representation of v is available. We relax this requirement in Section 4.

Convex budget sets. If the consumer faces a new budget constraint that is piece-wise linear, these two conditions lend themselves to a seemingly-simple iterative procedure for determining exact consumer welfare changes. The logic is easiest to describe in the case of an increasing price schedule, which implies a convex budget set.

An increasing price schedule is described by a set of marginal prices $p_1 < ... < p_{J+1}$ and quantities $0 < k_1 < k_2 < ... < k_J$ ("kink points") at which the prices increase. Starting with the lowest marginal price (i.e., $j = 1$), we calculate the 'virtual' compensating income $\hat{y}_j$ that solves

$$v(p_j, \hat{y}_j) = u^0 \ . \tag{1}$$

We then check if the corresponding demand is feasible given the nonlinear price schedule, by evaluating whether or not

$$k_{j-1} < q(p_j, \hat{y}_j) < k_j \ . \tag{2}$$

There are three cases to consider: If (2) holds, the minimum compensating income yields consumption at price p_j under the new schedule. If the right-hand inequality fails, then $\hat{y}_j$ is insufficient compensating income and we proceed to the next marginal price and again use (1). If the left-hand inequality fails, the minimum compensating income yields consumption at the previous kink point, k_{j-1}.

Depending which of these three cases obtains, determining the compensating variation is either simple or difficult. The simple situation occurs if (2) holds (strictly) for some j. Then the compensating variation is obtained by subtracting $\hat{y}_j$ from the (uncompensated) virtual income $y - d(p_j)$. The virtual income differs from initial income y due to the infra-marginal expenditure premium $d(p_j) = \sum_{i=1}^{j-1} k_i(p_i - p_{i+1})$ with nonlinear prices. The latter is easily obtained from the new price schedule.

The difficult situation occurs if the income-compensated demand occurs at a kink point. The problem here is that the marginal price at a kink point of the (new) budget constraint may not equal the consumer's marginal rate of substitution (MRS). It is the unknown MRS, not the marginal price, that is needed in (1) to determine the compensating income. In this case, the unknown MRS and compensating income are jointly determined by a pair of simultaneous nonlinear equations. The precise conditions are noted presently.

Non-convex budget sets. If a new budget set is non-convex, there may be more than one crossing of the new price schedule and the consumer's (compensated) demand function, as in Figure 3. This situation requires identifying each such crossing, and selecting the value of CV that yields the smallest compensating income.

With a piece-wise linear budget constraint, it is straightforward to find each crossing: For each p_j, a value of $\hat{y}_j$ that satisfies (1) and (2) yields a candidate value of $CV = y - d(p_j) - \hat{y}_j$. If there are no convex kinks in the budget constraint, the correct value of CV is the maximum of these candidate values (thus minimizing compensating income $y - CV$).

Summary. The basic logic in the existing literature amounts to the following procedure. Here we combine the convex and non-convex cases: The new budget constraint is piecewise linear with kink points $0 < k_1 < k_2 < ... < k_J$ and (possibly non-monotonic) intervening marginal prices $p_1, p_2, \ldots, p_{J+1}$.

Step 1. For each marginal price p_j, solve (1) for $\hat{y}_j$.

Step 2. For each pair $(p_j, \hat{y}_j)$ that satisfies (2), set $\widehat{CV}_j = y - d(p_j) - \hat{y}_j$. Retain the maximum of the $\widehat{CV}_j$'s. If $p_1 > p_2 > \ldots > p_{J+1}$ (no convex kinks), this max is the correct value of CV. Otherwise:

Step 3. For each price pair $p_j < p_{j+1}$ (convex kink) that satisfies

$$q(p_j, \hat{y}_j) \geq k_j \geq q(p_{j+1}, \hat{y}_{j+1}) \; , \tag{3}$$

find the 'virtual' price $p^* \in [p_j, p_{j+1}]$ and CV_j^* that solve

$$u^0 = v(p^*, y - d(p^*) - CV_j^*) \tag{4a}$$

$$k_j = q(p^*, y - d(p^*) - CV_j^*) \tag{4b}$$

where $d(p^*) = \sum_{i=1}^{j} k_i(p_i - p^*)$. Retain the maximum of the CV_j^*'s.

Step 4. The correct value of CV is the largest of the maxima from Steps 2 and 3.

Steps 1 and 2 are simple to implement; step 3 is not. Equation (3) selects each convex kink point k_j where demand crosses supply at a point of discontinuity of the price schedule. Equation (4a) then solves for the consumer's marginal rate of substitution (p^*) at precisely the level of compensating variation (CV_j^*) where the compensated demand equals the kink point quantity (4b). The main concern here is that (4a) and (4b) are a simultaneous system of nonlinear equations that can be difficult to solve for the compensating variation.[5]

Applications differ widely (or are altogether unclear) on implementations of this procedure. In early applications, Hausman (1983) and Blomquist (1983) simply assumed that consumption remains on the same budget segment j after price schedules change. DeBorger (1989) solves (4a)-(4b) numerically for an application of rationing, which presents a simple two-segment convex budget constraint. More recently, Fullerton and Gan (2004)

[5]With a convex budget set, at most one point will satisfy (3) so Step 3 must be completed (at most) once. In empirical models with unobserved heterogeneity in preferences, however, step 3 may need to be performed a large number of times to 'integrate out' the unobserved taste parameters that appear in $q(\cdot)$ and $v(\cdot)$. See Blundell et al. (1994) for an example.

suggest a way of handling kink points if the analytic form of the consumer's direct utility function is known. This simplifies the problem, but is of limited help when the direct utility function is not known. We suggest an alternative approach, which requires neither the direct nor indirect utility functions. It addresses the difficulties that arise at kinks by formulating the calculation of CV as a recursive system that is straightforward to solve and implement. This becomes enormously important for relaxing the presumption that v is known analytically.

Simplifications. A modification of the foregoing procedure handles kink points in a simpler manner using the compensated demand function, $h(p, u^0)$. The compensated demand is readily obtained from the indirect utility function v if the latter is known, and can be computed conveniently from the ordinary demand function if not. We assume for the moment h and v are known.

Step 1. For each p_j that satisfies $k_{j-1} < h(p_j, u^0) < k_j$, solve (1) for $\hat{y}_j$ as before.

Step 2. Set $\widehat{CV}_j = y - d(p_j) - \hat{y}_j$. Retain the maximum of the $\widehat{CV}_j$'s.

Step 3. For each price pair $p_j < p_{j+1}$ (convex kink) that satisfies

$$h(p_j, u^0) \geq k_j \geq h(p_{j+1}, u^0) \ , \tag{5}$$

find the 'virtual' price $p^* \in [p_j, p_{j+1}]$ for which

$$k_j = h(p^*, u^0) \ . \tag{6a}$$

Set $d(p^*) = \sum_{i=1}^{j} k_i(p_i - p^*)$ and solve for CV_j^*:

$$u^0 = v(p^*, y - d(p^*) - CV_j^*) \ . \tag{6b}$$

Retain the maximum of the CV_j^*'s.

Step 4. The correct value of CV is the largest of the maxima from Steps 2 and 3.

The key difference between this and the preceding algorithm occurs in Step 3. Equations (4a)-(4b) and (6a)-(6b) both evaluate the compensating variation if consumption occurs at a kink. However, (6a) and (6b) form a recursive system: they can be solved sequentially for the (unique) marginal rate of substitution p_j^* and compensating variation CV_j^* at each kink. This is particularly simple to perform in practice because h is monotonic in p^* and v is monotonic in CV^*. If the budget set is non-convex, these steps precisely parallel the graphical analysis of Figures 4(a)-(b) in Section 2.

General nonlinear price schedules. If a new price schedule $s(p)$ is not piecewise linear, the previous approach using the Hicksian demand is readily generalized. This generalization will prove useful to guide the analysis if indirect utility (and the Hicksian demand function) are unknown.

There are three necessary conditions for a value of CV to (just) compensate a consumer for a nonlinear budget constraint change:

$$s(p^*) = h(p^*, u^0) \qquad \textit{supply = demand} \qquad (7a)$$

$$d^* = P(s(p^*)) - p^* s(p^*) \qquad \textit{inframarginal premium} \qquad (7b)$$

$$u^0 = v(p^*, y - d^* - CV^*) \qquad \textit{indifference} \qquad (7c)$$

Here $P(q)$ denotes the consumer's total expenditure (i.e., the bill) for q units under the (new) nonlinear price schedule. The first equation determines the price(s) where the new supply schedule and demand intersect (as in Figure 3), and the second accounts for inframarginal prices that differ from the 'virtual' price p^* due to the nonlinearity of the price schedule. In budget-set space, equation (7a) defines a price vector p^* that (locally) separates the initial indifference curve and the new supply schedule, and (7c) determines the compensating variation that makes the new (compensated) budget set just touch this indifference curve.

A key feature of these equations is that they are recursive, and can be solved sequentially for p^*, d^*, and CV^*. Their solution poses familiar issues. When the supply schedule $s(p^*)$ is strictly increasing, the budget set is convex and (7a) has a unique solution for p^*. This price determines d^* and the compensating variation, CV^*.

If the supply schedule is decreasing there may be multiple solutions to (7a), as illustrated in Section 2. Each of these marginal prices may yield a different infra-marginal premium d^* and value of CV^*. Again, we choose the maximum of the candidate values of CV^* that satisfy (7a)-(7c) in order to award the smallest compensating income, $y - CV$. In practice, this means a researcher will have to determine at which price(s) compensated demand crosses supply, as in Figures 4(a)-(b). This can be done rapidly using the ordinary demand function q even if the Hicksian demand function h not analytically available, as shown next.

4 Exact Consumer Welfare Without Indirect Utility

The procedures discussed heretofore do not require knowledge of the consumer's direct utility function, only indirect utility. However, a demand system that fits the data well in applied work may be difficult, if not impractical, to solve analytically for indirect utility. Vartia (1983) proposed a method to evaluate exact consumer welfare using only the consumer's ordinary (Marshallian) demand function. His method, based on Sheppard's Lemma and other arguments, assumes a linear budget constraint. Suitable modifications enable one to evaluate exact consumer welfare changes with nonlinear prices quite generally.

As in the previous section, we use the three conditions in (7). The immediate problem is how to evaluate (7a) and (7c) when h and v are unknown. As in Vartia (1983), our approach is to find and evaluate the expenditure function as we move along the indifference surface that maintains initial utility.

Assuming the consumer faces a linear budget constraint, Sheppard's Lemma relates the expenditure function to the ordinary demand function via the differential equation

$$\frac{\partial e(p,u)}{\partial p} = q(p, e(p,u)) \tag{8}$$

with the boundary (initial) condition $e(p^0, u^0) = y$. Here q is a known function and e an unknown function to be solved. Integrating implies

$$e(p^*, u^0) - y = \int_{p^0}^{p^*} q(p, e(p, u^0)) \; dp \tag{9}$$

where p^* is the (unknown) marginal rate of substitution at the (to be determined) minimum compensating income. Again, this presumes a linear budget constraint.

With a nonlinear budget constraint, if we can evaluate p^* and $e(p^*, u^0)$ then we can determine CV by applying the recursion in (7). The central idea is to solve (9) for the unknown function $e(p, u^0)$ along a price path starting from p^0. Simultaneously, we determine p^* as the value(s) of p that set the integrand equal to the (new) nonlinear price schedule $s(p)$. This solves (7a). Given p^*, the inframarginal discount in (7b) is immediate from the price schedule. We then insert the value of $e(p^*, u^0) - y$ from (9) into (7c) to obtain a candidate compensating variation value for the nonlinear price schedule change:

$$CV^* = y - e(p^*, u^0) - d^* \; . \tag{10}$$

Note that if the consumer's initial (marginal) price p^0 is observed by the researcher, then it is not necessary to evaluate initial utility u^0 explicitly. There may be more than one (distinct) value of CV^* if the new budget set is non-convex. As before, we select the maximum of these CV^* values to award the minimum compensating income $y - CV$.

A variety of methods can be used to solve equation (9) for the unknown function

$e(p, u^0)$. Varitia (1983) proposed an Euler-Cauchy method due to Collatz (1963).[6] More sophisticated techniques are both more accurate and simple to implement. Imagine partitioning an interval $[p^0, p^1]$ that includes all possible values of p^* into a uniform grid: $p_n = p^0 + \frac{n}{N}(p^1 - p^0)$, $n = 0, 1, 2, \ldots, N$. Discretized, (9) becomes

$$\sum_{n=1}^{N} \left[e(p_n, u^0) - e(p_{n-1}, u^0)\right] = \sum_{n=1}^{N} \int_{p_{n-1}}^{p_n} q(p, e(p, u^0))\ dp \tag{11}$$

A standard (Runge-Kutta) method approximates each integral in (11) with a weighted average of two endpoint and two midpoint values of the integrand on $[p_{n-1}, p_n]$:

$$e_n - e_{n-1} = \tfrac{1}{6}\left[q_{n-1}^{(1)} + 2q_{n-1}^{(2)} + 2q_{n-1}^{(3)} + q_{n-1}^{(4)}\right] \times (p_n - p_{n-1})\ , \tag{12}$$

where

$$q_{n-1}^{(i)} = q\left(p_{n-1} + \delta a^{(i)}, e_{n-1} + \delta a^{(i)} q_{n-1}^{(i-1)}\right)$$

with $\delta = p_n - p_{n-1}$ and $a^{(1)} = 0$, $a^{(2)} = a^{(3)} = \frac{1}{2}$, $a^{(4)} = 1$. The starting value is $e_0 = y$. The value of e_n approximates $e(p_n, u^0)$ with an error of $O(\delta^4)$ as $\delta \downarrow 0$.[7]

It remains to determine the price(s) p^* at which to evaluate the candidate compensating variation values satisfying (10). This is done by checking when compensated demand equals supply, or

$$q(p_n, e_n) = s(p_n)$$

and then setting $p^* = p_n$. In practice this may not be satisfied exactly on a discrete price grid, so it is preferable to assign to p^* the value of $(p_n + p_{n-1})/2$ (or another interpolant) whenever the sign of the excess demand function $q(p_n, e_n) - s(p_n)$ changes. Note this will identify the value(s) of p^* that satisfy (7a) whether at a (convex) kink of the budget constraint or at an actual marginal price of the nonlinear price schedule, as required.

Summary. Assume the researcher observes the consumer's initial (marginal) price p^0 and income y, and the ordinary demand function $q(p, y)$ is known. Then the compensating variation for a new nonlinear price schedule $s(p)$ can be determined as follows:

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[6] Vartia's method approximates the integral in (11) with the average of its values at each endpoint:

$$e_n - e_{n-1} = \tfrac{1}{2}\left[q(p_n, e_n) + q(p_{n-1}, e_{n-1})\right] \times (p_n - p_{n-1})$$

Each value of e_n requires iterating on i

$$e_n^{(i)} = e_{n-1} + \tfrac{1}{2}[q(p_n, e_n^{(i-1)}) + q_{n-1})] \times (p_n - p_{n-1})$$

until $|e_n^{(i)} - e_n^{(i-1)}|$ is negligible, then proceeding to $n + 1$.

[7] The approximation in (12) is essentially a fourth-order expansion about q_{n-1}. More sophisticated implementations, such as using adaptive step sizes and progressive error-correction, are widely available for solving the difference equation in (11). Press et al. (2007) provide a useful survey.

Step 1. Choose prices $\underline{p}, \overline{p}$ that bound the marginal prices of the new price schedule;

Step 2. Define the interval(s) on which to evaluate $e(p, u^0)$:

(a) If $p^0 \leq \underline{p},$ set $I^1 = [p^0, \overline{p}]$ *(higher new prices)*

(b) If $\overline{p} \leq p^0,$ set $I^1 = [\underline{p}, p^0]$ *(lower new prices)*

(c) If $\underline{p} < p^0 < \overline{p},$ set $I^1 = [\underline{p}, p^0], I^2 = [p^0, \overline{p}]$ *(both higher and lower prices)*

Step 3. Break interval I^1 into a discrete grid of points. At each price point p_n evaluate e_n using (12) (or a similar approximation method).

Step 4. For each n such that

$$\text{sign}\left[q(p_n, e_n) - s(p_n)\right] \neq \text{sign}\left[q(p_{n-1}, e_{n-1}) - s(p_{n-1})\right],$$

set $p^* = \frac{1}{2}(p_n + p_{n-1})$, $e^* = \frac{1}{2}(e_n + e_{n-1})$, and $q^* = q(p^*, e^*)$. Let $d^* = P(q^*) - p^*q^*$ and

$$\widehat{CV}^* = y - d^* - e^* \ .$$

Retain the maximum of the $\widehat{CV}^*$'s.

Step 5. If step 2(c) applies, repeat steps 3 and 4 for the second interval I^2. The correct value of CV is the largest of the maxima from step 4.

We illustrate these ideas with two examples.

5 Examples

5.1 Volume Discounts

Volume discounts of the type depicted in Figure 2 are common for a wide array of products and services, including retail consumer goods, building materials, software licenses, car and hotel rental rates, and so on. As a simple example of the various methods discussed for calculating compensating (or equivalent) variation, consider Figure 5. It shows the compensated demand curve corresponding to the ordinary demand function

$$\begin{aligned} q(p, y) &= \alpha + \beta p + \gamma y \\ &= 64 - 5p + 0.1y \end{aligned} \tag{13}$$

for $y = 200$. Facing the initial price schedule $s^0(p)$ in Figure 5, the consumer initially chooses $q^0 = 9$ units at a marginal price of $p^0 = 15$.

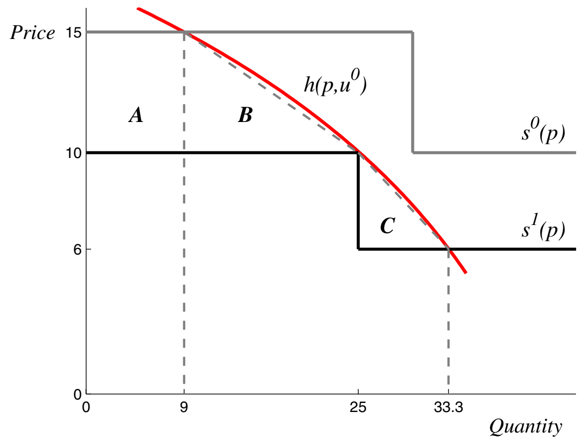

Figure 5

Known Indirect Utility

Graphically. Following the approach in Section 2, we need the area to the left of the Hicksian demand between the initial price $p^0 = 15$ and new marginal price $p^* = 6$. Given the slight curvature of the compensated demand function, we approximate CV by the sum of the three polygons A, B, and C. This yields a CV of approximately $101 = 45 + 40 + 16$. As the figure suggests, this value will slightly understate the compensating variation.

Analytically. To compute CV analytically, we require the indirect utility function corresponding to the demand function (13). Following Hausman (1981), we use Roy's identity to obtain

$$v(p, y) = \exp(-\gamma p)(y + \tfrac{1}{\gamma}(\alpha + \beta p + \beta/\gamma)) \tag{14}$$

which implies an initial utility level $u^0 = v(15, 200) \doteq -91.4834$. Now we use the three conditions in (7a)-(7c).[8] Equation (7a) yields a unique solution $p^* = 6$ and (7b) yields $d^* = 100$. At the new marginal price, (7c) implies CV satisfies

$$u^0 = \exp(-0.1 \times 6)(200 - CV - d^* + 10(64 - 5 \times 6 - 5/.1).$$

Solving gives $CV \doteq 106.6936$. Note the recursive structure of (7a)-(7c) greatly simplifies this calculation.

[8] To obtain the Hicksian demand function, invert indirect utility to obtain the expenditure function $e(p, u) = u \exp(\gamma p) - \frac{1}{\gamma}(\alpha + \beta p + \beta/\gamma)$. Differentiating with respect to p yields the compensated demand function $h(p, u) = \gamma u \exp(\gamma p) - \beta/\gamma$. In Figure 5, $h(p, u^0) \doteq -9.1483\ \exp(0.1p) - 50$.

TABLE 1
COMPENSATING VARIATION USING VARTIA'S METHOD

Step Size	$h(p, u^0)$ $p = 10$	Candidate CV	$h(p, u^0)$ $p = 6$	Candidate CV	Percent Error
1.0	25.1426	88.5738	33.3432	106.5684	0.1173
0.1	25.1324	88.6765	33.3308	106.6923	0.0012
0.01	25.1322	88.6776	33.3307	106.6935	1.2×10^{-5}
0.001	25.1322	88.6776	33.3306	106.6936	1.2×10^{-7}

Unknown Indirect Utility

Here we use methods in Section 4 to solve for CV. Because the marginal prices in the new schedule ($p_H^1 = 10$ and $p_L^1 = 6$) are below the initial marginal price ($p^0 = 15$), we consider a grid of prices between $p_L^1 = 6$ and $p^0 = 15$ (that is, step 2(b) applies). This interval includes p_H^1 so that we can determine whether or not the compensated demand function crosses the new supply schedule $s^1(p)$ at any quantity less than the threshold (kink point) at $k_1 = 25$.

Table 1 displays the results using Vartia's (Euler-Cauchy) method for step 3. The rows of the table correspond to the width of the interval between p_n and p_{n+1}. A size of 0.01, for example, solves a discrete approximation to (11) for prices 0.01 apart.[9]

The second column of Table 1 reports the compensated demand at the higher marginal price $p_H^1 = 10$. This exceeds 25 units, which is the maximum quantity available at this price with the new schedule. Hence, we know $p_* = p_L^1$. (The third column further reveals that the CV associated with consumption at $p_H^1 = 10$ is always less than the CV associated with consumption at the lower marginal price, $p_L^1 = 6$.)

Table 1 reveals that the accuracy of the Euler-Cauchy method is sensitive to the step size. Indeed, the last column suggests the percentage error decreases linearly with step size. Solving the the difference equation (in step 3) using the four-point Runge-Kutta method delivers much smaller approximation errors for the same grid values. With a step size of 1 the approximation error of Runge-Kutta is 0.0001 percent, and with a step size of 0.001 the error is 10^{-13} percent.

5.2 Menus of Prices

In many settings, consumers face not a single nonlinear price schedule but rather a *menu* of pricing plans. Common examples include mobile phone service plans, membership levels at museums and nonprofit organizations, and deductible choices for insurance. Menus of prices can present the consumer with an effective schedule of marginal prices that is not

[9]Each step in Vartia's approximation is iterated until $|e_n - e_{n-1}| < 10^{-9}$.

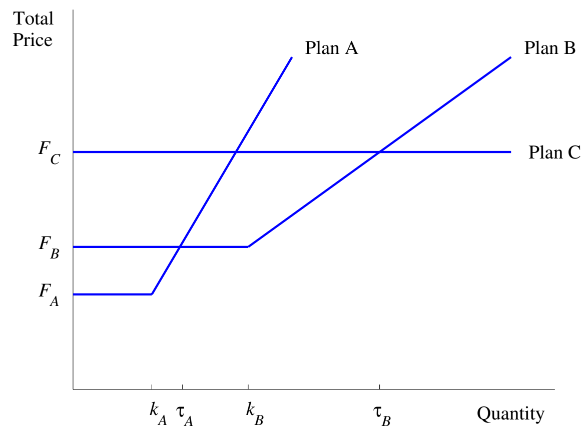

Figure 6

monotonic in quantity. Here we illustrate how the methods of Section 3 can be applied when consumers face a menu of prices.

Consider the example in Figure 6. Here the consumer faces a choice of one of three pricing plans, where each solid line in Figure 6 represents a different plan. Note that in this figure the vertical axis measures each plan's total price, *not* the marginal price. Plan A has a minimum fee F_A, allows up to k_A units to be consumed at a marginal price per unit of zero, and then imposes a positive marginal price p_A for each unit beyond k_A. Plan B has a similar structure, but with higher minimum fee F_B, a larger allowance k_B, and a lower marginal price p_B beyond this allowance. Plan C has a simpler structure: A higher minimum fee F_C than the other plans, but a marginal price of zero for all units consumed. As Figure 6 suggests, Plan C appeals to relatively high demand consumers, Plan B to moderate demand consumers, and Plan A to consumers with the lowest demand for this service.

This pricing scheme is used by a major European internet service provider for internet access in the residential market (quantity is in megabits downloaded per month). Lambrecht, Seim, and Skiera (2005) present an empirical analysis of consumer demand facing this price menu. The welfare question that arises in this setting is how consumers fare under this discriminatory form of internet service pricing, which differs from the flat-fee system prevalent in the United States.

Simplifying Lambrecht *et al.* slightly, assume that the consumer's ordinary demand for this service—*if* available at a constant marginal and average price p — again takes the linear form of the previous example. Now consider how to evaluate the change in income that would leave the consumer indifferent if its initial pricing plan was replaced by the menu of plans in Figure 6. Suppose that the initial pricing is a simple fixed fee, F, with no additional charge per unit used. In this case the initial marginal price is $p^0 = 0$, and the

TABLE 2
COMPENSATING VARIATION WITH A MENU OF PRICES

Plan	Quantity Interval	Marginal Price	Income-compensated Demand, $h(p, u^0)$	Excess Demand or Supply?
A	$0 < q \leq k_A$	0	$\gamma u^0 - \beta/\gamma$	*Excess demand:* $h(0, u^0) > k_A$
	$k_A < q \leq \tau_A$	p_A	$\gamma u^0 - \beta/\gamma - \exp(\gamma p_A)$	*Excess demand:* $h(p_A, u^0) > \tau_A$
B	$\tau_A < q \leq k_B$	0	$\gamma u^0 - \beta/\gamma$	*Excess demand:* $h(0, u^0) > k_B$
	$k_B < q \leq \tau_B$	p_B	$\gamma u^0 - \beta/\gamma - \exp(\gamma p_B)$	*Neither;* $k_B < h(p_B, u^0) \leq \tau_B$
C	$\tau_B < q$	0	$\gamma u^0 - \beta/\gamma$	*Excess supply:* $h(0, u^0) < \tau_B$

consumer's initial utility level is readily obtained from (14) using the fact that $d(p^0) = F$:

$$u^0 = (y - F) + \tfrac{1}{\gamma}(\alpha + \beta/\gamma). \tag{15}$$

To calculate the compensating variation with the menu of pricing plans, it remains to find the new marginal price p^* where $h(p, u^0)$ intersects supply. We assume the consumer chooses the expenditure-minimizing new plan that achieves utility u^0. Because the *marginal* prices (the slopes along the lower envelope) of the pricing plans in Figure 6 vary non-monotonically and discontinuously, we show the calculations in a table. The five rows in Table 2 partition all possible quantities into intervals according to plan and marginal price. These five intervals are indicated in the second column of Table 2 and in Figure 6, in terms of the allowance quantities k_A, k_B, and the threshold quantities τ_A, τ_B that separate expenditure-minimizing plans.

The fourth column of Table 2 shows the expenditure-minimizing consumption level that achieves u^0, but assuming each interval's marginal price is constant for all units consumed. This corresponds to evaluating (7a) at each marginal price among all plans offered.

To determine the new price p^* where (compensated) demand intersects supply, we check each interval (row in Table 2) to see if demand exceeds supply or supply exceeds demand. This is done in the last column of the table; it evaluates whether the quantity demanded (in column 4) falls within the admissible interval in column 2. Here we assume this occurs in fourth interval (row), which implies $p^* = p_B$.

The compensating variation now follows immediately from (the inverse of) (7c), using (14):

$$CV = y - u^0 \exp(\gamma p_B) + \tfrac{1}{\gamma}\left(\alpha + \beta p_B + \beta/\gamma\right) - [F_B - k_B p_B].$$

The final term in square brackets is the difference in inframarginal expenditure, d^*, with

Plan B relative to a constant (marginal and average) price p_B:

$$d^* = [F_B + (q - k_B)p_B] - q\,p_B = F_B - k_B\,p_B.$$

Although these calculations might appear a formidable procedure at first, Table 2 is computationally simple to implement and evaluate. The same procedure can be easily extended to situations where the price structure has more marginal prices, or more plans are offered, and so on.

6 Conclusion

This paper proposes a simplified method for evaluating exact consumer welfare when consumers face nonlinear prices, including menus of pricing plans. Our treatment covers the case where the researcher knows indirect utility as well as when it is not analytically available. We also show these methods can easily handle non-convex budget sets, which are otherwise considered to pose difficult problems in the literature. We hope these methods simplify future research.

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