The Partial Naivete Euler Equation

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Abstract

This short paper extends the Harris and Laibson (2001) heuristic derivation of the Hyperbolic Euler Equation to the partial naivete setting. The result is used to show that if the coefficient of relative risk aversion exceeds one, saving is increasing in sophistication.

JEL codes: D91 (Intertemporal Consumer Choice; Life Cycle Models and Saving), E21 (Consumption; Saving; Wealth)

1 Introduction

A growing literature examines the consequences of naive quasi-hyperbolic discounting for a wide range of economic behavior.[1] O'Donoghue and Rabin (1999a, 1999b) demonstrate that procrastination is predicted by models of naivete. DellaVigna and Malmendier (2004) and Eliaz and Spiegler (2004) analyze the design of contracts when agents may be naive, finding similarities to contracts observed in a range of industries. Fang and Silverman (2004) and Paserman (2004) demonstrate that the naivete model helps to explain the duration of welfare and unemployment spells, respectively. Shui and Ausubel (2004) estimate a model using data from a randomized field experiment on credit card take-up and use. Excess sensitivity to teaser rates relative to post-teaser rates provides evidence for naivete. Skiba and Tobacman (2006) use payday loan borrowing, repayment, and default data to identify naivete. In all these contexts, procrastination opportunities distinguish naivete.

This note studies identification of naivete with *standard* consumption data, showing that for typical (known) values of the coefficient of relative risk aversion, naifs and sophisticates will have similar consumption paths and *not* be

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[1] Introduced by Strotz (1955) and Akerlof (1991), naive quasi-hyperbolic discounting differs from sophisticated quasi-hyperbolic discounting (Phelps and Pollak 1968, Laibson 1997) in the assumption that the current self believes future selves will implement the current self's preferences.

distinguishable. This result accounts for the simulation results of Angeletos, Laibson, Repetto, Tobacman and Weinberg (2001). Section 2 presents the central theoretical result, the Partial Naivete Euler Equation, and Section 3 uses it to assess the effect of sophistication on consumption. Section 4 briefly concludes.

2 Main Result

Let X_t, c_t, and Y_t equal cash-on-hand, consumption, and income, respectively, at time t. Assume that self t's utility is given by[2]

$$\begin{aligned} U_t\left(X_t\right) &= u\left(c_t\right) + \beta\delta E_t\left[V_{t+1}(X_{t+1})\right], \text{ where} \\ V_t(X_t) &= u\left(c_t^*\right) + \delta E_t\left[V_{t+1}(X_{t+1}\left(c_t^*\right))\right], \text{ and} \\ c_t^* &= \arg\max U_t\left(X_t\right) \text{ subject to the dynamic budget constraints} \\ X_{t+1} &= R(X_t - c_t) + Y_{t+1} \end{aligned}$$

Assume, moreover, that V is everywhere differentiable.[3]

Proposition 1 *Let β^E be the expected short-term discount factor of future selves, and let $\frac{\partial c_{t+1}^E}{\partial X_{t+1}}$ be the associated expected MPC. Then*

$$u'\left(c_t\right) = E_t R\left[\beta\delta\frac{\partial c_{t+1}^E}{\partial X_{t+1}} + \frac{\beta}{\beta^E}\delta\left(1 - \frac{\partial c_{t+1}^E}{\partial X_{t+1}}\right)\right]u'\left(c_{t+1}^E\right),$$

where c_{t+1}^E is the level of consumption self t (erroneously) expects self $t+1$ to choose.

Proof #1: Dynamic Programming. Following the "heuristic derivation" of Harris and Laibson (2001), note that the identity

$$U_t\left(X_t\right) = \beta V_t\left(X_t\right) + (1-\beta)\,u\left(c_t\right)$$

links U_t and V_t. Maximizing $U_t\left(X_t\right)$ over c_t, the first order condition reads:

$$u'\left(c_t\right) = \beta\delta R E_t\left[V_{t+1}'(X_{t+1})\right],$$

and the Envelope Theorem implies

$$U_t'\left(X_t\right) = u'\left(c_t\right)$$

[2] This recursive representation derives naturally from representations of total utility as an infinite sum.

[3] This assumption is not trivial: strategic interactions between the various temporal selves can induce non-monotonicities in the consumption function and non-differentiability of the value function. Harris and Laibson (2001) show that such poor behavior is smoothed out when income distributions have dense, sufficiently large support.

Differentiate the identity linking U_t and V_t with respect to X_t to find

$$U_t'(X_t) = \beta V_t'(X_t) + (1-\beta)\, u'(c_t)\, c_t'(X_t)$$

Insert the envelope theorem into this expression and isolate V_t'. Then we obtain

$$V_t'(X_t) = \frac{1}{\beta}\left[1-(1-\beta)\, c_t'(X_t)\right] u'(c_t)$$

Now shift all of these variables forward one period. Since the consumer believes that one period in the future decisions will be made with one-period discount factor β^E rather than β, insert superscript E's accordingly.

$$V_{t+1}'(X_{t+1}) = \frac{1}{\beta^E}\left[1-\left(1-\beta^E\right) c_{t+1}^{E\prime}(X_{t+1})\right] u'\left(c_{t+1}^E\right)$$

Finally, combine this last expression with the FOC to arrive at the result:

$$u'(c_t) = E_t R\left[\beta\delta \frac{\partial c_{t+1}^E}{\partial X_{t+1}} + \frac{\beta}{\beta^E}\delta\left(1-\frac{\partial c_{t+1}^E}{\partial X_{t+1}}\right)\right] u'\left(c_{t+1}^E\right)$$

■

Proof #2: Sequence Problem. The Hyperbolic Euler Equation reads:

$$u'(c_t) = E_t R\left[\beta\delta c_{t+1}'(X_{t+1}) + \delta\left(1-c_{t+1}'(X_{t+1})\right)\right] u'(c_{t+1})$$

Under perfect certainty, in an infinite-horizon eat-the-pie problem, if preferences are CRRA, and if wealth at time t is given by w_t, the constant λ such that $c_t = \lambda w_t$ is given by[4]

$$\lambda = 1-\left\{R^{1-\rho}\delta\left[\beta\lambda + (1-\lambda)\right]\right\}^{1/\rho} \tag{1}$$

In the partial naivete case, when the agent believes future selves have the short-term discount factor $\beta^E > \beta$, she believes future selves will consume according to the rule $c_t = \lambda^E w_t$, where λ^E is given analogously by:

$$\lambda^E = 1-\left\{R^{1-\rho}\delta\left[\beta^E\lambda^E + \left(1-\lambda^E\right)\right]\right\}^{1/\rho} \tag{2}$$

[4]I.e., λ characterizes the unique Markov Perfect Equilibrium of the infinite horizon game between the agent's various temporal selves.

Given these beliefs,

$$
\begin{aligned}
c_{t+i}^{E} &= (w_t - c_t)\,\lambda^{E} R^{i} \left(1-\lambda^{E}\right)^{i-1}, \text{ and} \\
U_t &= u(c_t) + \beta \sum_{i=1}^{\infty} \delta^{i} u\left(c_{t+i}^{E}\right) \\
&= \frac{c_t^{1-\rho}}{1-\rho} + \frac{\beta}{1-\rho} \sum_{i=1}^{\infty} \delta^{i} \left[(w_t - c_t)\,\lambda^{E} R^{i} \left(1-\lambda^{E}\right)^{i-1}\right]^{1-\rho} \\
&= \frac{c_t^{1-\rho}}{1-\rho} + \frac{\beta\delta \left[(w_t - c_t)\,\lambda^{E} R\right]^{1-\rho}}{1-\rho} \sum_{j=0}^{\infty} \left[\delta \left(R\left(1-\lambda^{E}\right)\right)^{1-\rho}\right]^{j} \\
&= \frac{c_t^{1-\rho}}{1-\rho} + \frac{\beta\delta \left[(w_t - c_t)\,\lambda^{E} R\right]^{1-\rho}}{1-\rho} \left(\frac{1}{1-\left[\delta \left(R\left(1-\lambda^{E}\right)\right)^{1-\rho}\right]}\right)
\end{aligned}
$$

The first order condition with respect to c_t reads:

$$
\begin{aligned}
c_t^{-\rho} &= \left(\frac{\beta\delta R\lambda^{E}}{1-\left[\delta \left(R\left(1-\lambda^{E}\right)\right)^{1-\rho}\right]}\right) \left[(w_t - c_t)\,\lambda^{E} R\right]^{-\rho}, \text{ or} \\
u'(c_t) &= \left(\frac{\beta\delta R\lambda^{E}}{1-\left[\delta \left(R\left(1-\lambda^{E}\right)\right)^{1-\rho}\right]}\right) u'\left(c_{t+1}^{E}\right)
\end{aligned}
$$

Rearrangement using the (implicit) definition of λ^{E} yields a perfect-certainty version of the Partial Naivete Euler Equation:

$$
\begin{aligned}
u'(c_t) &= \frac{\beta}{\beta^{E}} \delta R \left[\beta^{E}\lambda^{E} + \left(1-\lambda^{E}\right)\right] u'\left(c_{t+1}^{E}\right) \\
&= R\left[\beta\delta\lambda^{E} + \frac{\beta}{\beta^{E}}\delta\left(1-\lambda^{E}\right)\right] u'\left(c_{t+1}^{E}\right)
\end{aligned}
$$

■

Several remarks on this result are in order.

Remark 1 *When $\beta^{E} = \beta$ and $\lambda^{E} = \lambda$, we recover the Harris and Laibson (2001) Hyperbolic Euler Equation,*

$$u'(c_t) = E_t\left\{R\left[\beta\delta\lambda + \delta(1-\lambda)\right] u'(c_{t+1})\right\}$$

Remark 2 *When $\beta^{E} = 1$, we obtain*

$$
\begin{aligned}
u'(c_t) &= \beta\delta R E_t u'\left(c_{t+1}^{E}\right), \quad \textit{where } c_{t+1}^{E} \textit{ solves} \\
u'\left(c_{t+1}^{E}\right) &= \delta R E_t u'\left(c_{t+2}^{E}\right)
\end{aligned}
$$

This means the agent behaves like an exponential discounter with a one-period discount factor of $\beta\delta$, with respect to the consumption levels erroneously anticipated in future periods.

Remark 3 *Intuitively, at time t the consumer expects the MPC at time $t+1$ to be λ^E. When she expects that a marginal dollar will be spent at time $t+1$, she applies the discount factor $\beta\delta$ as in the perfect sophistication case. However, when she expects a marginal dollar to be saved at time $t+1$, she applies a discount factor of $\frac{\beta}{\beta^E}\delta$, which we typically think is less than δ. This captures the idea that as β^E increases the consumer thinks she is* particularly impatient right now.

Remark 4 *More specifically, as β^E increases the consumer thinks the only relevant tradeoff is between today and next period, because future selves will act increasingly in the interest of the current self. In the extreme case where $\beta^E = 1$, future selves will behave exactly correctly (the current self expects) so the only tradeoff that matters is between today and next period. In that case, the one-period discount factor $\beta\delta$ is all that matters.*

3 The Effect of Sophistication

The effect of sophistication on consumption is also of interest. In general there are two offsetting effects. Increasing naivete means increasing one's faith in future selves' patience – faith that when assets are transferred to the future, future selves will transfer the assets still further forward in time. This effect promotes saving. Second, by contrast, raising naivete also raises expectations that one's future selves will save more on their own, regardless of one's current behavior. This effect discourages saving. The following Proposition characterizes the net effect.

Proposition 2 *(i) If preferences are CRRA with coefficient of relative risk aversion of $\rho = 0, 1$, or $\to \infty$, then saving does not depend on the degree of sophistication. (ii) If $\rho > 1$, saving is increasing in sophistication (equivalently, current consumption is increasing in naivete, ie, $\frac{d\lambda}{d\beta^E} > 0$), and in a neighborhood to the left of $\rho = 1$ saving is decreasing in sophistication.*

Proof of (i). When $\rho = 0$, no saving ever occurs, when $\rho = 1$ the two effects exactly offset, and in the limit as $\rho \to \infty$ then everything is always saved.

In general, if we consider the linear consumption rule $c_t = \lambda w_t$, then self t expects consumption of $c_{t+1} = (1-\lambda) R w_t \lambda^E$ in period $t+1$. This implies that in the perfect certainty case we can rewrite the Euler equation and solve for λ

as follows:

$$
\begin{aligned}
\left[\frac{(1-\lambda) R w_t \lambda^E}{\lambda w_t}\right]^\rho &= \frac{\beta}{\beta^E}\delta R\left[\beta^E\lambda^E + \left(1-\lambda^E\right)\right] \\
\left[\frac{R\lambda^E}{\lambda} - R\lambda^E\right]^\rho &= \beta\delta R\left[\lambda^E + \frac{\left(1-\lambda^E\right)}{\beta^E}\right] \\
\left[\frac{1}{\lambda} - 1\right] &= \frac{1}{\lambda^E R}\left\{\beta\delta R\left[\lambda^E + \frac{\left(1-\lambda^E\right)}{\beta^E}\right]\right\}^{1/\rho} \\
\implies & \boxed{\lambda = \left\{1 + \tfrac{1}{\lambda^E R}\left\{\beta\delta R\left[\lambda^E + \tfrac{\left(1-\lambda^E\right)}{\beta^E}\right]\right\}^{1/\rho}\right\}^{-1}} \qquad (3)
\end{aligned}
$$

Note that an implicit expression defining λ^E is given in Equation 2. In the special case $\rho = 1$,

$$\lambda^E = \frac{1-\delta}{1-\left(1-\beta^E\right)\delta}, \quad \text{and hence}$$

$$
\begin{aligned}
\lambda &= \left\{1 + \beta\delta\left[\frac{\lambda^E\beta^E + \left(1-\lambda^E\right)}{\lambda^E\beta^E}\right]\right\}^{-1} \\
&= \left\{1 + \frac{\beta\delta}{\lambda^E\beta^E}\left[\frac{\beta^E(1-\delta)}{1-\left(1-\beta^E\right)\delta} + \frac{\beta^E\delta}{1-\left(1-\beta^E\right)\delta}\right]\right\}^{-1} \\
&= \left\{1 + \frac{\beta\delta}{\lambda^E\beta^E}\left[\frac{\beta^E}{1-\left(1-\beta^E\right)\delta}\right]\right\}^{-1} \\
&= \left\{1 + \frac{\beta\delta}{1-\delta}\right\}^{-1} \\
&= \frac{1-\delta}{1-(1-\beta)\delta}
\end{aligned}
$$

which is independent of λ^E and β^E. Thus when $\rho = 1$ the degree of sophistication has no effect on saving. ■

Proof of (ii). More generally, we have to differentiate Equation 3 for λ with respect to β^E and evaluate the derivative's sign as a function of ρ. First, substitute Equation 2 into the Partial Naivete Euler Equation to obtain

$$\beta^E\left(\frac{\lambda^E}{1-\lambda^E}\right)^\rho = \beta\left(\frac{\lambda}{1-\lambda}\right)^\rho,$$

which implies

$$\ln\left(\frac{\lambda^E}{1-\lambda^E}\right)-\ln\left(\frac{\lambda}{1-\lambda}\right)=\frac{1}{\rho}\left(\ln\beta-\ln\beta^E\right)$$

These are all monotone increasing transformations, proving that

$$\lambda^E>\lambda\forall\beta^E<\beta$$

Now define $\Lambda\equiv\ln\left(\frac{\lambda}{1-\lambda}\right)$, $\Lambda^E\equiv\ln\left(\frac{\lambda^E}{1-\lambda^E}\right)$, $B=\ln\beta$, and $B^E=\ln\beta^E$. Then we see

$$\Lambda-\Lambda^E=-\frac{1}{\rho}\left(B-B^E\right),$$

implying

$$\frac{\partial\Lambda}{\partial B^E}=\frac{1}{\rho}+\frac{\partial\Lambda^E}{\partial B^E},$$

where the second term can be computed using

$$\frac{\partial\Lambda^E}{\partial B^E}=\frac{\partial\lambda^E}{\partial\beta^E}\frac{\beta^E}{\lambda^E\left(1-\lambda^E\right)}$$

and differentiating Equation 2 wrt β^E. We eventually obtain

$$\frac{\partial\Lambda}{\partial B^E}=\frac{\left(1-\lambda^E\right)\left(1-\beta^E\right)(1-\rho)}{\left(1-\lambda^E\right)\left(1-\beta^E\right)(1-\rho)-\beta^E\rho}$$

This expression demonstrates again that at $\rho=1$ consumption is independent of the degree of naivete. In addition, $\rho>1$ clearly implies this expression is positive.

When $\rho<1$, the expression still cannot be signed trivially: though the numerator is positive, the sign of the denominator depends on how quickly λ^E rises as ρ falls. It is obvious, however, that $\frac{\partial\Lambda}{\partial B^E}$ must be negative in a neighborhood to the left of $\rho=1$. ■

4 Conclusion

This note makes three contributions. First, it presents the Partial Naivete Euler Equation. Second, complementing existing studies that estimate models of naivete, it indicates why naivete and sophistication may be difficult to distinguish just with data on consumption. Third, the paper shows that naivete increases the MPC if the coefficient of relative risk aversion exceeds 1.

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